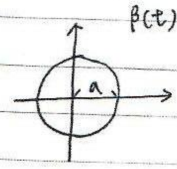
<u>Def</u>: A parametriged Curve is a map. $\alpha: I \subseteq \mathbb{R} \to \mathbb{R}^3$
of an open interval $I = (a,b)$ into $\mathbb{R}^3$
$\alpha(t) = (x(t), y(t), z(t))$
$I = (a,b)$ can be the whole line (ie. $a \to -\infty$, $b \to +\infty$)
Image of $\alpha \equiv C \equiv \{x(t), y(t), z(t) \mid t \in I\}$ is called the trace of curve.

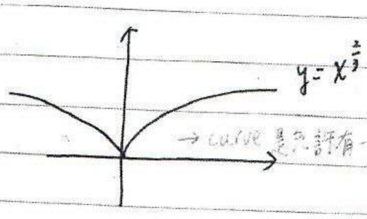
<u>Def</u>: A parametrized differentiable curve is a differentiable map $\alpha: I \to \mathbb{R}^3$
α is differentiable (C^∞) iff x, y, z are C^∞
In general, $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the tangent vector (velocity vector) of the curve of t.

<u>Ex</u>: $\alpha(-\varepsilon, 2\pi+\varepsilon) \to \mathbb{R}^2$
$\alpha(t) = (a\cos t, a\sin t)$, $a > 0$.
$\beta(t) = (a\cos 2t, a\sin 2t)$



$a\cos t$ and $a\sin t$ is diff $\to$ $\alpha(t)$ diff
不同的curve，但有相同的trace

<u>Ex</u>: $\alpha(t) = (t^3, t^2)$
$\alpha: \mathbb{R} \to \mathbb{R}^2$
$x(t) = t^3$, $y(t) = t^2$ $\to$ smooth
$\alpha(t)$ is the graph of $y = x^{\frac{2}{3}}$
$\alpha(0) = (0,0)$



→ curve 是允許有一個 corner

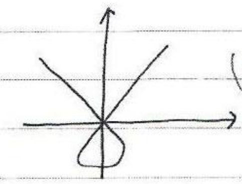
Ex: $\alpha: I \to \mathbb{R}^2$
$\alpha(t) = (t, |t|)$
α is not a parametrized differentiable curve

Note: A curve describe the motion of a partical in $\mathbb{R}^3$ and trace is the trajectory but with differential speed or direction, the curve is consider to be different

Ex: $\alpha: \mathbb{R} \to \mathbb{R}^2$

$(\cos t, \sin t) \neq (\cos 2t, \sin 2t)$

$\alpha(t) = (t^3 - t,\ t^2 - 1)$



$\alpha(1) = (0,0)$

$\alpha(-1) = (0,0)$

$\Rightarrow$ p.d curve

f is 1-1, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

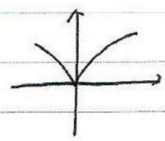
$\therefore \alpha$ is not requrred to be an 1-1 function.

Def: A parametrized differential curve $\alpha: I \to \mathbb{R}^3$ is regular if, $\alpha'(t) \neq 0,\ \forall t \in I$

Ex: $\alpha(t) = (t^3, t^2)$

$\alpha'(t) = (3t^2, 2t)$

$\alpha'(t)\big|_{t=0} = (0,0)$



α is not a regular curve

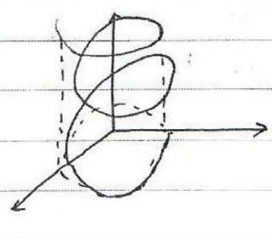
Ex: (helix)

$\alpha: \mathbb{R} \to \mathbb{R}^3$ is given by $\alpha(t) = (\underbrace{a\cos t}_{C^\infty}, \underbrace{a\sin t}_{C^\infty}, \underbrace{bt}_{C^\infty})$, $a \neq 0$, $b \neq 0$

$\alpha(0) = (a, 0, 0)$

$\alpha\left(\frac{\pi}{2}\right) = \left(0, a, \frac{\pi b}{2}\right)$

$\alpha(\pi) = (a, 0, b)$



$\alpha(t)$ has its trace in $\mathbb{R}^3$

a helix of pictch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$.

$\Rightarrow \alpha(t)$ is C^∞

$\alpha'(t) = (-a\sin t, a\cos t, b)$

$\alpha'(t) \neq 0$ for all $t \in \mathbb{R}$

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Def: The arc-length of a regular curve $\alpha: I \to \mathbb{R}^3$
to a point $t_0 \in I$ is defined by $s(t) = \int_{t_0}^{t} |\alpha'(s)| ds = \int_{t_0}^{t} \sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}\, ds$

recall:

Def: The arc-length of a regular curve $\alpha: I \to \mathbb{R}^3$ to a point $t_0 \in I$
is defined by $s(t) = \int_{t_0}^{t} |\alpha'(s)| ds = \int_{t_0}^{t} \sqrt{(x'(s))^2 + (y'(s))^2 + (z'(s))^2}\, ds$

- $\because \alpha'(t) \neq 0, \forall t \in I$

① The arc-length $s(t)$ is a differentiable function.

② By Fundamential Theorem of calculus, $\frac{ds}{dt} = |\alpha'(t)|$

③ It depends on the image of α, not α

recall: $|\alpha'(t)|^2 = \langle \alpha'(t), \alpha'(t) \rangle = x'(t)^2 + y'(t)^2 + z'(t)^2$

- Inner product

$\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

$\vec{u} \cdot \vec{v} = (u_1v_1 + u_2v_2 + u_3v_3) = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$

- wedge product (Cross Product)

$$\vec{u} \wedge \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$|\vec{u} \wedge \vec{v}| = |\vec{u}||\vec{v}||\sin\theta|$

Ex:

$\alpha(t) = (a\cos t, a\sin t)$, $t \in [0, 2\pi]$

$\alpha'(t) = (-a\sin t, a\cos t)$, $|\alpha'(t)| = a$

$S = \int_0^{2\pi} |\alpha'(t)|\, dt = 2\pi a$

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Def: A regular curve $\alpha(s)$ is parametrized by arc-length if $|\alpha'(s)| = 1$

Fact: Every regular curve can be reparametrized by arc-length

Idea: Find $t(s)$ + $\alpha(s)$ satisfies $|\alpha'(\boxed{t(s)})| = 1$

$$\frac{d\alpha(t(s))}{ds} = \frac{d\alpha}{dt}\frac{dt}{ds} = 1 \Rightarrow \left|\frac{d\alpha}{dt}\right|\left|\frac{dt}{ds}\right| = 1 \Rightarrow \left|\frac{d\alpha}{dt}\right| = \left|\frac{ds}{dt}\right|$$

We may assume $\frac{dt}{ds} \geq 0$

$\int \left|\frac{d\alpha}{dt}\right| dt = \int \frac{ds}{dt} dt = s(t)$

$\because$ α is regular $\therefore$ $s(t)$ is well defined $\Rightarrow$ $t(s)$ is exists.

Ex: ① straight line

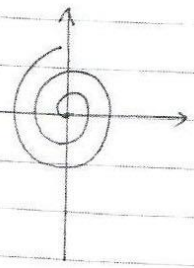
$\alpha(t) = \vec{a}(t) + \vec{b}$, where $\vec{a}, \vec{b}$: non-zero constant vector

$\alpha'(t) = \vec{a}$, $s(t) = \int_0^t |\alpha'(u)|\, du = \|\vec{a}\|\, t$

$\Rightarrow$ α can be reparametrized by arc-length

$\because$ $t(s) = \frac{s}{\|\vec{a}\|}$, $\alpha(s) = \frac{\vec{a}}{\|\vec{a}\|} s + \vec{b}$

② $\alpha : \mathbb{R} \to \mathbb{R}^2$ logarithmic spiral



$\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$

$\to \alpha'(t) = (ae^{bt}(b\cos t - \sin t), ae^{bt}(b\sin t + \cos t))$

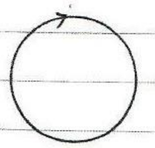
$s = \int_0^t |\alpha'(u)|\, du = \int_0^t \sqrt{a^2e^{2bu}[b^2(\cos^2 u + \sin^2 u) + (\cos^2 u + \sin^2 u)]}\, du$

$= \int_0^t \sqrt{a^2e^{2bu}(b^2+1)}\, du$

$\to s = \frac{a\sqrt{b^2+1}}{b}(e^{bt} - 1)$ $\Rightarrow$ $t = \frac{1}{b}\ln\left(\frac{bs}{a\sqrt{b^2+1}} + 1\right)$

$\alpha(t) = \vec{a}t + \vec{b}$

$\alpha'(t) = \vec{a} \quad , \forall t$



$\alpha(t) = (\cos t, \sin t)$

$\alpha'(t) = (-\sin t, \cos t)$

tangent vector always change

- Let $\alpha(s)$ be a curve parametrized by arc length

  $\frac{d}{ds}(\alpha(s)) = \alpha'(s) = \vec{t}$ (tangent vector)

  $(\vec{t}\,') = \alpha''(s) \rightarrow$ measure the change of the tangent vector at s , $|\vec{t}|^2 = 1$

$|\vec{t}| = 1$, $2\langle \vec{t}, \vec{t}\,' \rangle = 0 \Rightarrow \vec{t} \perp \vec{t}\,'$

$\vec{t}\,' = k\vec{n}$, n is the normal vector to tangent vector $\vec{t}$ and parallel to $(\vec{t})'$

$\vec{n} = \frac{\vec{t}\,'}{k}$ is well-define if $k \neq 0$.

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Def: $k(s) = |\alpha''(s)|$ is called the curvature of the curve α at s

$k(s) \geq 0$.

From now no, $k(s) > 0$.

Def: A plane is determined by $\vec{t}(s)$ and $\vec{n}(s)$ is called osculating plane at s.

Ex: ① A straight $\alpha(s) = \vec{a}s + \vec{b}$

$\alpha'(s) = \vec{a}$, $|\alpha'(s)| = |\vec{a}| = 1$

$\alpha''(s) = 0$, $k(s) \equiv 0$.

② $\alpha(t) = (a\cos t, a\sin t)$, $a > 0$, $a \neq 1$

$\alpha'(t) = (-a\sin t, a\cos t)$, $|\alpha'(t)| = a$

$$s = \int_0^t |\alpha'(u)|\, du = at \Rightarrow t = \frac{s}{a}$$

$\alpha(s) = \left(a\cos\left(\frac{s}{a}\right), a\sin\left(\frac{s}{a}\right)\right) \longrightarrow$ New parametrization

$|\alpha'(s)| = 1$

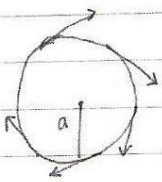
$\alpha'(s) = \left(-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)\right)$

$\alpha''(s) = (\vec{t}') = \left(\frac{-1}{a}\cos\left(\frac{s}{a}\right), -\frac{1}{a}\sin\left(\frac{s}{a}\right)\right)$

$|\vec{t}'| = |\alpha''(s)| = \frac{1}{a} = k(s)$

$\vec{n} = \left(-\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right)\right)$ $\left(\because \vec{n} = \frac{\vec{t}'}{k}\right)$

curvature of circle (radius a)



Def: Let $\alpha(s)$ be a curve parametrized by arc-length s.t $k(s) > 0$, $\forall s \in I$

$\vec{b} = \vec{t} \wedge \vec{n}$ is called binomal vector. to α at s.

$\{\vec{t}, \vec{n}, \vec{b}\}$ is called Frenet frame (trihedrm)

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Plane Curves

curvature $k = \dfrac{\alpha'(s)}{|\alpha'(s)|}$

<u>Def</u>: Let $\alpha(s)$ be a curve parametrized by arc-length s.t

$k(s) > 0 \ \forall s \in I$

$\vec{b} = \vec{t} \wedge \vec{n}$ is called binomal vector to α at s.

$\{\vec{t}, \vec{n}, \vec{b}\}$ is called Frenet frame.

$|\vec{b}| = |\vec{t} \wedge \vec{n}| = |\vec{t}||\vec{n}| \sin\frac{\pi}{2} = 1$

$\vec{b}$ is perpendicular to osculaty plane (spanned by $\vec{t}$ and $\vec{n}$)

$\vec{b}'$ measures how the osculativy plane is moving

$\vec{b}' = (\vec{t} \wedge \vec{n})' = \vec{t}' \wedge \vec{n} + \vec{t} \wedge \vec{n}' = k\cancel{\vec{n} \wedge \vec{n}}^{\,0} + \vec{t} \wedge \vec{n}' = \underline{\vec{t} \wedge \vec{n}'}$ —① $\therefore \vec{b}' \perp \vec{t}$

$|\vec{b}|^2 = 1$, $\langle \vec{b} \cdot \vec{b} \rangle = 1 \to 2\langle \vec{b}' \cdot \vec{b} \rangle = 0 \ \therefore \vec{b}' \perp \vec{b}$ —②

$\therefore \vec{b}'$ is parallel to $\vec{n}$ $\ni \vec{b}' = \square\, \vec{n}$

<u>Def</u>: Let $\alpha(s)$ be a curve parametrized by arc-length s.t $\alpha''(s) \neq 0$ $\forall s \in I$. The number $\tau(s)$ defined by $\vec{b}' = \tau(s)\vec{n}(s)$ is called the torsion of α at s. (扭力)

Now, we have $\vec{t}' = k\vec{n}$, k: curvature

$\vec{b}' = \tau \vec{n}$, τ: torsion

$$\vec{n}' = ? \quad \begin{cases} \langle \vec{n} \cdot \vec{t} \rangle = 0 \\ \langle \vec{n} \cdot \vec{b} \rangle = 0 \end{cases} \Rightarrow \begin{cases} \langle \vec{n}' \cdot \vec{t} \rangle = -\langle \vec{n} \cdot \vec{t}' \rangle = -k|\vec{n}|^2 = -k(s) \\ \langle \vec{n}' \cdot \vec{b} \rangle = -\langle \vec{n} \cdot \vec{b}' \rangle = -\tau(s) \end{cases}$$

$$\begin{cases} \vec{t}' = k\vec{n} \\ \vec{n}' = -k\vec{t} - \tau\vec{n} \\ \vec{b}' = \tau\vec{n} \end{cases}$$ is call Frenet Formula

recall: $\{u, v, w\}$ are positively oriented iff $\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} > 0$,

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$

$\vec{b} = \vec{t} \wedge \vec{n}$

$\vec{t} \to \vec{n} \to \vec{b} \to \vec{t}$

$\because \vec{n} = \vec{b} \wedge \vec{t}$

$$(\vec{n}') = (\vec{b}') \wedge \vec{t} + \vec{b} \wedge \vec{t}'$$
$$= \tau(s)\vec{n} \wedge \vec{t} + \vec{b} \wedge (k(s)\vec{n})$$
$$= -\tau(s)\vec{b} + -k(s)\vec{t}$$

Def: A plane spanned by $\vec{n}$ and $\vec{b}$ is called normal plane

Def: A plane spanned by $\vec{b}$ and $\vec{t}$ is called rectifying plane

Ex: $\alpha(t) = (a\cos t, a\sin t, bt)$

$\alpha(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c})$, where $c = \sqrt{a^2+b^2}$ ← parametrized by arc-length

$\alpha'(s) = \left(\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) = \vec{t}$

$|\alpha'(s)| = \sqrt{\frac{a^2+b^2}{c^2}} = 1$

$\alpha''(s) = \left(\frac{-a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right) = \vec{t}'$

$k(s) = |\alpha''(s)| = \sqrt{\frac{a^2}{c^4}} = \frac{a}{c^2}$

$\vec{n} = \frac{\alpha''(s)}{|\alpha''(s)|} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$

$$\vec{b} = \vec{t} \wedge \vec{n} = \begin{vmatrix} i & j & k \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{vmatrix} = \left(+\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right)$$

$\because \vec{b}' = \tau(s)\vec{n}$, $\vec{b}' = \left(\frac{b}{c^2}\sin\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0\right) \Rightarrow \tau = -\frac{b}{c^2} = \frac{-b}{a^2+b^2}$

helix $\Rightarrow$ 考慮 $b = 0 \Rightarrow k = \frac{1}{a}$, $\tau = 0$.

circle

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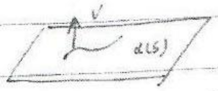
• **Thm**: A curve in $\mathbb{R}^3$ which has $K>0$ is a plane curve iff $\tau \equiv 0$.

pf: "$\Rightarrow$"

Let $\alpha : I \to \mathbb{R}^3$ is a plane curve, Given any constant vectors β, r

s.t $\alpha(s)\cdot\beta = r$



$(\alpha(s)-\alpha(s_0))\cdot V = 0$

$(\alpha(s)\cdot\beta)' = (r') = 0$

$\therefore \alpha'(s)\cdot\beta = 0 \Rightarrow \vec{t}\cdot\beta = 0,\ \vec{t}\perp\beta$

$(\alpha'(s)\cdot\beta)' = 0 \Rightarrow \alpha''(s)\beta + \alpha'(s)\cdot\overset{0}{\beta'} = 0$

$\Rightarrow \alpha''(s)\beta = 0 \Rightarrow \vec{n}\cdot\beta = 0,\ \beta\perp\vec{n}$

$\vec{\beta} \parallel \vec{b},\ \vec{b} = c\vec{\beta} \Rightarrow \vec{b}' = 0 \Rightarrow \tau = 0$

"$\Leftarrow$"

$\because \tau = 0,\ \vec{b}' = 0 \Rightarrow \vec{b} \equiv$ constant vector

$(\alpha(s)\cdot\vec{b})' = \alpha'(s)\cdot\vec{b} + \alpha(s)\cdot\overset{0}{\vec{b}'} = \vec{t}\cdot\vec{b} = 0$

$\alpha(s)\cdot\vec{b} =$ const $\therefore \alpha(s)$ is a plane curve.

※※※※※ HomeworK ※※※※※

① Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc-length

$K = 0$ if and only if α is a piece of a line.

② Let $\alpha : I \to \mathbb{R}^3$ is a curve parametrized by arc-length and $\alpha(s)$ has $K>0$ and $\tau = 0 \Rightarrow \alpha(s)$ is a part of a circle of radius $\frac{1}{K}$

$\alpha(t) = (t, t^2, t^3)$

$\alpha'(t) = (1, 2t, 3t^2)$, $S = \int_0^t \sqrt{1+4u^2+9u^4}\, du$

plance curve → 多直, 多彎 → K

space curve → τ

Find curvatural and torsion τ without assuming α is parametrized by arc-length. α is a regular curve, i.e. $\alpha'(t) \neq 0$, $\forall t \in I$.

$\frac{d\alpha}{ds} = \frac{d\alpha}{dt} \cdot \frac{dt}{ds}$. Since α can be parametrized by arc-length.

$$1 = \left|\frac{d\alpha}{ds}\right| = \underbrace{\left|\frac{d\alpha}{dt}\right|}_{|\alpha'|} \frac{dt}{ds} \Rightarrow \frac{dt}{ds} = \frac{1}{|\alpha'|} \quad — ①$$

$$\vec{t} = \alpha' \frac{dt}{ds}$$

$$\vec{t}' = \frac{d}{ds}\left(\alpha' \frac{dt}{ds}\right) = \frac{d}{dt}(\alpha') \frac{dt}{ds} \cdot \frac{dt}{ds} + \alpha' \frac{d^2t}{ds^2}$$

$$= \alpha'' \left(\frac{dt}{ds}\right)^2 + \alpha' \frac{d^2t}{ds^2}$$

$$k\vec{n} = \frac{\alpha''}{|\alpha'|^2} + \alpha' \frac{d^2t}{ds^2} \quad — ②$$

$$② \cdot \alpha' \Rightarrow 0 = kn \cdot \alpha' = \frac{\alpha'' \cdot \alpha'}{|\alpha'|^2} + |\alpha'|^2 \frac{d^2t}{ds^2}$$

$$\Rightarrow \frac{d^2t}{ds^2} = \frac{-\alpha'' \cdot \alpha}{|\alpha'|^4} \quad — \text{plag into ②}$$

∴ ② can be rewritten as

$$\vec{t}'' = k\vec{n} = \frac{\alpha''}{|\alpha'|^2} + \alpha' \cdot \frac{-\alpha'' \cdot \alpha'}{\|\alpha'\|^4} \quad — ③$$

$$k = \sqrt{\left(\frac{\alpha''}{|\alpha'|^2}\right)^2 + \left(\alpha' \frac{(-\alpha'' \cdot \alpha)}{|\alpha'|^4}\right)^2 - 2\frac{(\alpha'' \cdot \alpha')^2}{|\alpha'|^6}} = \sqrt{\frac{|\alpha''|^2}{|\alpha'|^4} - \frac{(\alpha'' \cdot \alpha')^2}{|\alpha'|^6}}$$

∵ $(\alpha'' \cdot \alpha')^2 = |\alpha''|^2 |\alpha'|^2 \cos^2\theta$, $K = \sqrt{\frac{|\alpha''|^2 |\alpha'|^2 (1-\cos^2\theta)}{|\alpha'|^6}}$

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$$k = \sqrt{\frac{|\alpha''\wedge\alpha'|^2}{|\alpha'|^6}} = \frac{|\alpha''\wedge\alpha'|}{|\alpha'|^3} \quad —— (*)$$

$$\vec{n} = \frac{|\alpha'|\,\alpha''}{|\alpha'\wedge\alpha''|} - \frac{(\alpha''\cdot\alpha')\alpha'}{|\alpha'|\,|\alpha'\wedge\alpha''|} = \frac{|\alpha'|^2\alpha'' - (\alpha''\cdot\alpha')\alpha'}{|\alpha'|\,|\alpha'\wedge\alpha''|}$$

$$\vec{b} = \vec{t}\wedge\vec{n} = \frac{\alpha'}{|\alpha'|}\wedge\left(\frac{|\alpha'|^2\alpha'' - (\alpha''\cdot\alpha')\alpha'}{|\alpha'|\,|\alpha'\wedge\alpha''|}\right)$$

$$= \frac{\alpha'\wedge\alpha''}{|\alpha'\wedge\alpha''|}$$

$$\vec{b}' = \frac{|\alpha'\wedge\alpha''|\,(\underbrace{\alpha''\wedge\alpha''}_{0} + \alpha'\wedge\alpha''') - (\alpha'\wedge\alpha'')\left(\sqrt{(\alpha'\wedge\alpha'')(\alpha'\wedge\alpha'')}\right)'}{|\alpha'\wedge\alpha''|^2}$$

$$= \frac{\alpha'\wedge\alpha'''}{|\alpha'\wedge\alpha''|} - \frac{(\alpha'\wedge\alpha'')\left[(\alpha'\wedge\alpha''')\cdot(\alpha'\wedge\alpha'')\right]}{|\alpha'|\,|\alpha'\wedge\alpha''|^3}$$

$$\vec{b}' = \tau\vec{n}$$

$$\tau = \vec{b}'\cdot\vec{n} = \frac{(\alpha'\wedge\alpha''')\cdot\alpha''}{|\alpha'\wedge\alpha''|^2} \quad —— (*)(*)$$

$\alpha(t) = (a\cos t, a\sin t, bt)$, $a>0$, $b<0$.

Method I. find K & τ

$s = ct$, $t = \frac{s}{c}$, $K = \frac{a}{a^2+b^2}$, $\tau = \frac{-b}{a^2+b^2}$

Method II. use $*$ and $**$

$\frac{d\alpha}{dt} = (-a\sin t, a\cos t, b)$

$|\alpha'| = \sqrt{a^2+b^2}$, $\frac{dt}{ds} = \frac{1}{\sqrt{a^2+b^2}}$

HW ① $\alpha(t) = (t, t^2, t^3)$, $t \in \mathbb{R}^3$ → twisted cubic

② $\alpha(t) = (\cos t,\ 2(1-\sin t),\ \sqrt{3}\cos t)$

$$\frac{d\alpha}{ds} = 1 \Rightarrow \frac{dt}{ds} = \frac{1}{|\alpha'|} = \frac{1}{\sqrt{1+4t^2+9t^4}}$$

$\alpha'' = (0, 2, 6t)$

$\alpha' = (1, 2t, 3t^2)$

$\alpha''' = (0, 0, 6)$

$$\vec{t} = \alpha' \cdot \frac{dt}{ds} = \frac{1}{\sqrt{1+4t^2+9t^4}}(1, 2t, 3t^2)$$

$$\vec{t}' = \frac{(0,2,6t)}{1+4t^2+9t^2} - (1,2t,3t^2) \cdot \frac{4t+18t^3}{(1+4t^2+9t^2)^2}$$

$$K = \frac{|(-6t^2, +6t, -2)|}{\sqrt{1+4t^2+9t^4}^{\,3}} = \frac{2\sqrt{9t^4+9t^2+1}}{\sqrt{9t^4+4t^2+1}^{\,3}}$$

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Ex: Let $\alpha(t) = (x(t), y(t))$, $\alpha: I \to \mathbb{R}^2$

$\alpha'(t) = (x'(t), y'(t))$

$$\vec{t} = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{(x'(t), y'(t))}{\sqrt{x'^2(t) + y'^2(t)}}$$

$$\vec{t}' = \frac{\alpha''}{|\alpha'|^2} + \frac{\alpha'(-\alpha''\cdot\alpha')}{|\alpha'|^4} = \frac{(x'', y'')}{x'^2+y'^2} - \frac{(x', y')\big((x'', y'')\cdot(x', y')\big)}{(x'^2+y'^2)^2}$$

$$= \frac{(x'', y'')}{x'^2+y'^2} - \frac{(x'x''+y'y'')(x', y')}{(x'^2+y'^2)^2}$$

$\vec{t}' = k\vec{n}$

$\because$ on $\mathbb{R}^2$, $\vec{t} \perp \vec{n}$

$$\vec{n} = \frac{(-y'(t), x(t))}{\sqrt{x'^2(t) + y'^2(t)}} \Rightarrow k = \frac{-x''y' + y''x'}{(x'^2+y'^2)^{\frac{3}{2}}}$$

on $\mathbb{R}^3$, $k > 0$.

Def: A rigid motion in $\mathbb{R}^3$ is a result of composing a translation with an orthogonal transformation with positive determinant.

Recall: ① A translation by a vector $v \in \mathbb{R}^3$ is a map $A: \mathbb{R}^3 \to \mathbb{R}^3$ that is given by $A(p) = p + v$, $p \in \mathbb{R}^3$

② A linear map $\rho: \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation $\rho v \cdot \rho u = u \cdot v$, $\forall u, v \in \mathbb{R}^3$

Remark: arc-length, curvature and torsion are invariant under rigid motion $M: \mathbb{R}^3 \to \mathbb{R}^3$ rigid motion

$\alpha(t)$ a curve

$$\bar{\alpha}(t) = M \circ \alpha = \underset{\uparrow\ \text{rotation}}{\rho} \circ \alpha + \underset{\uparrow\ \text{translation}}{C}$$

$$\int_a^b \left|\frac{d\alpha}{dt}\right| dt = \int_a^b \left|\frac{d\, M\circ\alpha}{dt}\right| dt$$

Thm: (Fundamental Theorem of the curve (local))

Given differential function $\kappa(s)>0$, $\tau(s)$, $s\in I$, $\exists$ a regular curve parametrized by arc-length $\alpha: I \to \mathbb{R}^3$, s is the arc-length of α, $\kappa(s)$ is the curvature of α and $\tau(s)$ is the torsion of α. Moreover, α is unique up to rigid motion.

i.e. $\exists$ rigid motion $M: \mathbb{R}^3 \to \mathbb{R}^3$ $\ni$ $\bar{\alpha} = M\circ\alpha = \overset{\text{rotation}}{\rho}\circ\alpha + C \to$ translation

$\bar{\alpha}$ has same s, $\kappa(s)$, $\tau(s)$ as α

pf:

"Uniquness"

Let $\alpha(s)$ and $\bar{\alpha}(s)$ be two curve parametrized by $s\in[a,b]$

$\to$ s is their arc-length, and $\kappa(s)$, $\tau(s)$, $\bar{\kappa}(s)$ and $\bar{\tau}(s)$ satisfy $\kappa(s)=\bar{\kappa}(s)$

$\tau(s)=\bar{\tau}(s)$

Let $s_0\in[a,b]$, consider $\beta(s)=\bar{\alpha}(s)-(\bar{\alpha}(s_0)-\alpha(s_0))$

$\beta(s_0)=\bar{\alpha}(s_0)-(\bar{\alpha}(s_0)-\alpha(s_0))=\bar{\alpha}(s_0)$

$\beta(s_0)\equiv$ a translation of $\alpha(s)$

Now rotate β + $\beta'(s_0)=\alpha'(s_0)$ $\to$ s_0上 tangent vector 相同

Call the resulting curve $r(s)$

rotate again to match the normal vector at s_0.

Now all the resulting curve $\theta(s)$

$\therefore$ $\theta(s)$, $\alpha(s)$ have identical κ and τ

and $\alpha(s_0)=\theta(s_0)$

$\alpha'(s_0)=\theta'(s_0)$

$\alpha''(s_0)=\theta''(s_0)$, and also with identical binomal vector at s.

Let $\{\vec{t},\vec{n},\vec{b}\}$ and $\{\vec{\bar{t}},\vec{\bar{n}},\vec{\bar{b}}\}$ be the Frenet frame for $\alpha(s)$ and $\theta(s)$.

$t(s_0)=\bar{t}(s_0)$

$n(s_0)=\bar{n}(s_0)$

$b(s_0)=\bar{b}(s_0)$

Consider $f(s) = |t(s)-\bar{t}(s)|^2 + |n(s)-\bar{n}(s)|^2 + |b-\bar{b}|^2$

$f'(s) = 2[\langle t-\bar{t}, t'-\bar{t}'\rangle + \langle n-\bar{n}, n'-\bar{n}'\rangle + \langle b-\bar{b}, b'-\bar{b}'\rangle]$

$= 2[\langle t-\bar{t}, kn-k\bar{n}\rangle + \langle n-\bar{n}, -kt-\tau b+k\bar{t}+\tau\bar{b}\rangle + \langle b-\bar{b}, \tau n-\tau\bar{n}\rangle]$

$= 2[k\langle t-\bar{t}, n-\bar{n}\rangle + -k\langle n-\bar{n}, t-\bar{t}\rangle - \tau\langle n-\bar{n}, b-\bar{b}\rangle + \tau\langle b-\bar{b}, n-\bar{n}\rangle]$

$= 0$

$\Rightarrow f'(s)=0 \Rightarrow f(s)$ constant

$f(s_0)=0 \Rightarrow f(s)\equiv 0, \ \forall s\in I=[a,b]$

$\Rightarrow t=\bar{t}, \ n=\bar{n}, \ b=\bar{b} \quad \forall t\in I$

$g(s)=|\alpha(s)-\theta(s)|^2$

$g'(s) = 2\langle 2\alpha(s)-Q(s)\ \alpha'(s)-\theta'(s)\rangle$

$= 2\langle \alpha(s)-\theta(s), t-\bar{t}\rangle \underset{\text{by } ①}{=} 0$

$\forall s, \ g'(s)=0.$

However $g(s_0) = \underline{\alpha s_0 - \theta s_0} = |\alpha(s_0)-\theta(s_0)|^2$

$\Rightarrow g(s)\equiv 0 \Rightarrow \alpha(s)\equiv Q(s), \ \forall s\in I$

"existence"

we want to construct curve from $k(s)>0$. and $\tau(s)$, $t(s)=\alpha'(s)$

construct $\{t,n,b\}$ satisfy the Frent Formula

$$\begin{cases} t'=kn \\ b'=\tau n \\ n'=-kt-\tau b \end{cases}$$

if we set $t=(t_1,t_2,t_3)$, $n=(n_1,n_2,n_3)$

$$* \begin{cases} t_1'=kn_1 \\ t_2'=kn_2 \\ t_3'=kn_3 \end{cases} \qquad n_1'=-kt_1-\tau b_1 \qquad b_1'=\tau n_1$$

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There are nine equations . * has 9×9 unknow

* is 9×9 linear ODE

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0_{3\times3} & k I_{3\times3} & 0_{3\times3} \\ -k I_{3\times3} & 0_{3\times3} & -\tau_{3\times3} \\ 0_{3\times3} & \tau I_{3\times3} & 0_{3\times3} \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

where $0_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

By O.D.E, Theory, if we presuike the volume of t, n, p at a

fixed point, then ∃ solution to the system

We take $t(s_0) = (1,0,0)$, $n(s_0) = (0,1,0)$ and $b(s_0) = (0,0,1)$

If $h = [a,b] \to \mathbb{R}^9$.

$h(s) = (t(s), n(s), b(s))$

$h(s_0) = (e_1, e_2, e_3)$

$\therefore \begin{cases} h'(s) = A(s)\, h(s) \\ h(s_0) = (100, 010, 001) \end{cases}$

By ODE theory theorem, there exist a unique solution.

$g: [a,b] \to \mathbb{R}^9$ to the initial value problem

No claim that $\{t, n, b\}$ are othogonal

① $\begin{cases} |t^2| = 1 & \langle t, n \rangle = 0 \\ |n^2| = 1 & \langle n, b \rangle = 0 \\ |b^2| = 1 & \langle b, t \rangle = 0 . \end{cases}$

$(|t|^2)' = 2\langle t', t\rangle = 2k\langle t, n\rangle$

$(|n^2|)' = 2\langle n', n\rangle = -2k\langle n, t\rangle - 2\tau\langle n, b\rangle$

$(|b|^2)' = 2\cdot\langle b', b\rangle = 2\tau\langle b, n\rangle$

$\langle t, n\rangle' = \langle t', n\rangle + \langle t, n'\rangle = k|n|^2 - k|t^2| - 2\ \langle t, b\rangle$

$\langle n, b\rangle' = \langle n', b\rangle + \langle n, b'\rangle = 2|n|^2 - k\langle b, t\rangle - \tau|b|^2$

$\langle t, b\rangle' = k\langle n, b\rangle + \tau\langle t, n\rangle$

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l vector valued function

$l : [a, b] \to \mathbb{R}^6$

$l(s) = (|t(s)|^2, |n(s)|^2, |b(s)|^2, \langle t, n\rangle, \langle n, b\rangle, \langle t, b\rangle)$

$$◎ \begin{cases} \frac{dl}{ds} = M(s)\, l(s) \\ l(s)\big|_{s=s_0} = (1,1,1,0,0,0) \end{cases} \quad \left(\because \begin{array}{ll} |t(s_0)|^2 = 1, & \langle t, n\rangle|_{s=s_0} = 0 \\ |b(s_0)|^2 = 1 & \langle n, b\rangle|_{s=s_0} = 0 \\ |n(s_0)|^2 = 1 & \langle t, b\rangle|_{s=s_0} = 0 \end{array} \right.$$

where $M(s) = \begin{pmatrix} 0 & 0 & 0 & 2K & 0 & 0 \\ 0 & 0 & 0 & -2K & -2\tau & 0 \\ 0 & 0 & 0 & 0 & 2\tau & 0 \\ -K & K & 0 & 0 & 0 & \tau \\ 0 & \tau & -\tau & 0 & 0 & -K \\ 0 & 0 & 0 & \tau & K & 0 \end{pmatrix}$

Consider constant curve $u(s) \equiv (1,1,1,0,0,0)$ is a solution. of ◎

($\because$ ◎ is a linear solution)

$\Rightarrow u(s) = l(s) \Rightarrow l(s) = (1,1,1,0,0,0)$

$\therefore$ we have $\begin{cases} |t|^2 = 1, & \langle t, b\rangle = 0 \\ |n|^2 = 1, & \langle t, n\rangle = 0 \qquad \forall s \in I \\ |b|^2 = 1, & \langle n, b\rangle = 0 \end{cases}$

$\{t, n, b\}$ satisfy the Frent Formula

$\alpha(s) = \int_{s_0}^{s} t(u)\,du = \int_{s_0}^{s} \alpha'(u)\,du$

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CH 2

1-7 Global Property of Plane Curve

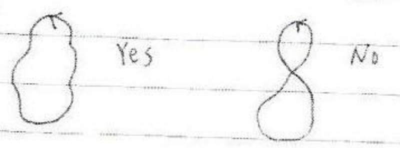
Def: A curve $\alpha : [a,b] \to \mathbb{R}^3$ is a $\boxed{\text{closed}}$ curve if $\begin{cases} \alpha(a) = \alpha(b) \\ \alpha'(a) = \alpha'(b) \\ \vdots \\ \alpha^{(n)}(a) = \alpha^{(n)}(b) \end{cases}$

α and its deritives agree at a and b

Def: A simple closed curve in $\mathbb{R}^3$ is a curve with no self-intersection and $\alpha(t_1) = \alpha(t_2)$ for some $t_1, t_2 \in [a,b]$, i.e $t_1 = t_2$

Def: A plane curve $\alpha : [a,b] \to \mathbb{R}^2$ is a positivity oriented simple closed curve if the intension region enclosed by curve is on the left hand side.

Ex:



Thm: Isoperimetric Inequality

Let $\alpha : [a,b] \to \mathbb{R}^2$ be a positive oriented simple closed curve in $\mathbb{R}^2$

Denote l = length of α

A = area of region enclosed (bounded) by α in $\mathbb{R}^2$

then $4\pi A \le \ell^2$, equality holds iff the image of α is circle

recall: Green's Thm

Let C be a positively oriented piecewise smooth, simple closed curve in $\mathbb{R}^2$, and let R be the region enclosed by C.

If f and g are functions (x,y) defined on open region containing R and have continuous partial derivative, then

$f,g \in C^1$

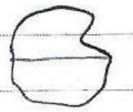
$$\int_C \left(f\frac{dx}{dt} + g\frac{dy}{dt}\right) dt = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$f=-y$, $g=x$: $\int_C (-yx' + xy')\,dt = \iint_R (1+1)\,dx\,dy = 2A(R)$

$\therefore$ Area$(R) = A = \frac{1}{2}\int_C (xy' - yx')\,dt$

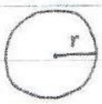
"pf":

$A \le \frac{\ell^2}{4\pi}$



diameter $= d = 2r$

$A_c = \pi r^2$



R = region enclosed by α

Let α curve be parametrized by arc-length

$\alpha(s) = (x(s), y(s))$, $s \in [0, \ell]$

Let curve β (circle) be parametrized by arc-length

$\beta(s) = (x(s), \bar{y}(s))$

$x^2(s) + \bar{y}^2(s) = r^2$

We know that $A(R)$ = area of region $R = \frac{1}{2}\int_0^\ell (xy' - yx')\,ds$

$\because \int_0^\ell (x'y)\,ds = (xy)\Big|_0^\ell - \int_0^\ell xy'\,ds$

$\therefore \int_0^\ell (x'y)\,ds = -\int_0^\ell xy'\,ds$

$\int_0^\ell x'y\,ds = -\int_0^\ell y'x\,ds$ — ②

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$A(R) = \frac{1}{2}\int_0^l (xy'-yx')ds = \frac{1}{2}\int_0^l (xy'+xy')\,ds = \int_0^l xy'\,ds$

$A(cir) = $ (intension of circle) $= \pi r^2$

$= \frac{1}{2}\int_0^l (x\bar{y}'-\bar{y}x')\,ds$

$= \frac{1}{2}\int_0^l (-x'\bar{y} + -x'\bar{y})\,ds$ > by ②

$= -\int_0^l (\bar{y}x')\,ds$

$A(R) + A(\text{circle}) = A(R) + \pi r^2 = \int_0^l (xy'-\bar{y}x')\,ds \le \int_0^l \sqrt{(xy'-\bar{y}x')^2}\,ds$

$\overset{***}{\le} \int_0^l \sqrt{(x^2+\bar{y}^2)\cdot(x'^2+y'^2)}\,ds = \int_0^l r\,ds = lr$ — **

"=" ⇒ R: a circle

We have

$A(R) + A(cir) \le lr$ — **

$\sqrt{A(R)}\,(\pi r^2)^{\frac{1}{2}} = \sqrt{A(R)}\sqrt{A(cir)} \le \frac{1}{2}(A(R)+A(cir)) = \frac{1}{2}(A(R)+\pi r^2) \le \frac{lr}{2}$

$\Rightarrow A(R)\cdot\pi r^2 \le \frac{l^2r^2}{4} \Rightarrow A(R) \le \frac{l^2}{4\pi}$

HW: hint ***

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② proof:

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recall: Isoperimetric Inequality

C: positively oriented. simple closed curve

R: region enclose by C

l: length of C

$$A(R) \le \frac{l^2}{4\pi}$$

Green's Theorem

$f \cdot g \in C^1(\mathbb{R}^2)$

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \int_C \left(f \frac{dx}{dt} + g\frac{dy}{dt}\right) dt$$

$$A(R) = \frac{1}{2}\int_0^l (-yx' + xy')\,ds.$$

$$A(R) = -\frac{1}{2}\int_0^l (\alpha \cdot \vec{n})\,ds.$$

$$A(R) \le \frac{1}{2}\int_0^l |\alpha|\,ds$$ (wirtiger Inequality)

Pressley (p.59-p.61)

Rmk: Isopermetric inegality also holds for α where α is piecewise continuous. allow finite # corner

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Def : A subset $S \subseteq \mathbb{R}^3$ is a regular surface iff all $p \in S$ $\exists$ a nbhd $V \subseteq \mathbb{R}^3$ and a map $\vec{x} : U \subseteq \mathbb{R}^2 \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ on to $V \cap S \subseteq \mathbb{R}^3$ s.t

① $\vec{x}$ is differentiable
i.e $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ all partial derivatives of $\vec{x}$ are continuous

② $\vec{x}$ is homeomorphism
$\vec{x}$: 1-1, onto, $(\vec{x})^{-1}$ exist and $(\vec{x})^{-1}$: continuous

③ $\vec{x}$: satisfies the nonsingular condition. i.e $\forall q \in U$,
the differentiable $d\vec{x} : \mathbb{R}^2 \to \mathbb{R}^3$ is 1-1
(We want the map $d\vec{x}$ takes the unit tangent in U into the tangent vector in $V \cap S$)

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The vector e_1 is tangent vector to the curve $M(u, V_0)$

The image curve $\alpha(u) = \vec{X}(u, V_0) = (x(u,V_0), y(u,V_0), z(u,V_0))$

$$\alpha'(u) = \frac{\partial x}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

Similary, take e_2 is tangent vector $\to (u_0, V)$

The image curve $B(V) = \vec{X}(u_0, V) = (x(u_0,V), y(u_0,V), z(u_0,V))$

$$B'(V) = \frac{\partial \vec{X}}{\partial V} = \left(\frac{\partial x}{\partial V}, \frac{\partial y}{\partial V}, \frac{\partial z}{\partial V}\right)$$

consider $r(t) = (at + u_0, bt + V_0)$

$r(0) = (u_0, V_0)$, $r'(t) = (a, b) = \omega$

The image curve $\beta(t) = \vec{X}(at+u_0, bt+V_0)$

$$\beta'(t) = \frac{\partial X}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial t} = a\frac{\partial X}{\partial u} + b\frac{\partial y}{\partial V}$$

$$d x_q(ae_1 + be_2) = a\frac{\partial X}{\partial u} + b\frac{\partial X}{\partial V} = a\, dX_q(e_1) + b\, dX_q(e_2)$$

$\therefore$ dX_q is a linear map.

$$d\bar{X}_q : \begin{bmatrix} \frac{dx}{\partial u} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial V} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial V} \end{bmatrix}$$

$d\bar{x}_q$ is injective (1-1) iff $\frac{\partial \bar{x}}{\partial u} \wedge \frac{\partial \bar{x}}{\partial v} \neq 0$

i.e at least one of minors of $A = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$

$\frac{\partial(x,y)}{\partial(u,v)}$ $\frac{\partial(y,z)}{\partial(u,v)}$ $\frac{\partial(x,z)}{\partial(u,v)}$ is non zero

where $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

$$\frac{\partial(x,y)}{\partial(\theta,\varphi)} = \det\begin{bmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi \end{bmatrix} = \cos\theta\sin\theta$$

$$\frac{\partial(y,z)}{\partial(\theta,\varphi)} = \sin^2\theta\cos\varphi \qquad ③\ \frac{\partial x}{\partial\theta} \wedge \frac{\partial x}{\partial\varphi} \neq 0$$

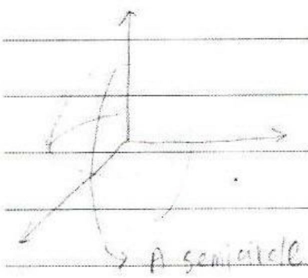
$$\frac{\partial(x,z)}{\partial(\theta,\varphi)} = -\sin^2\theta\sin\varphi$$

$\theta = \frac{\pi}{2}$, ① $= 0$, ② $\neq 0$, ③ $\neq 0$. $\therefore$ $d\tilde{x}$ is non-singular

rotate $180^\circ \to U' = \{(\theta,\varphi) \mid 0<\theta<\pi,\ 0<\varphi<\pi\}$ (2π?)

$\tilde{x} : U' \to \tilde{x}(U')$

$\underbrace{\vec{x}(U)}_{\text{regular}} \cup \underbrace{\tilde{x}(U')}_{\text{regular}} = \underbrace{S^2}_{\to \text{regular}} \Rightarrow S^2$: regular



Ex: $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

$\vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$.

$\vec{x}(u,v) = (u, v, \sqrt{1-u^2-v^2})$, $U = \{(u,v) \mid u^2+v^2<1\}$

① $\vec{x}$ is differentiable

$\frac{\partial x}{\partial u} = 1 \quad \frac{\partial y}{\partial u} = 0 \quad \frac{\partial z}{\partial u} = -u\sqrt{1-u^2-v^2}^{\,-1}$

$\frac{\partial x}{\partial v} = 0 \quad \frac{\partial y}{\partial v} = 1 \quad \frac{\partial z}{\partial v} = \frac{-v}{\sqrt{1-u^2-v^2}}$

② $\vec{x}$ is homeomorphism.

1-1, onto.

$\vec{x}(u,v) = \vec{x}(u',v') \Rightarrow \begin{cases} u'=u \\ v'=v \end{cases}$

$\vec{x}^{-1}$ exist

$\because \vec{x}^{-1}$ is the restriction of projection $(x,y,z) \to (x,y)$ to $\vec{x}(U)$)

$\vec{x}^{-1}$: continuous.

③ $\frac{\partial \vec{x}}{\partial u} \wedge \frac{\partial \vec{v}}{\partial v} \neq 0$ (?)

$\because \boxed{\frac{\partial(x,y)}{\partial(u,v)}} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$

$\therefore d\vec{x}$ is non-sigular.

$\vec{x}(U)$: upper hemisphere

$\vec{x}_2 : U \to \mathbb{R}^3$

$\vec{x}_2(u,v) = (u, v, -\sqrt{1-u^2-v^2}) \to$ lower hemisphere

$\vec{x}_1(U) \cup \vec{x}_2(U) = S^2 \setminus E$, $E = \{(x,y,0) \mid x^2+y^2=1\}$

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$\vec{X}_3(u,v) = (u, \sqrt{1-u^2-v^2}, v)$

$\vec{X}_4(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$

$\vec{X}_5(u,v) = (\sqrt{1-u^2-v^2}, u, v)$

$\vec{X}_6(u,v) = (-\sqrt{1-u^2-v^2}, u, v)$

$\vec{X}_1(U) \cup \vec{X}_2(U) \cup \vec{X}_3(U) \cup \vec{X}_4(U) \cup \vec{X}_5(U) \cup \vec{X}_6(U) = S^2$

$\Rightarrow S^2$: regular surface

prop: If $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a differential function on an open set $U \subseteq \mathbb{R}^2$ then the graph of f (i.e the set of pts in $\mathbb{R}^2$ given by $(x, y, f(x,y))\ \forall x, y \in U$) is a regular surface.

pf: We need to produce a regular parametrization of the surface S. is equal to the graph of f.

$\vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, $\vec{x}(u,v) = (u, v, f(u,v))$

① $\vec{x}$ is differentiable

② $\vec{x}$ is 1-1, onto, $\vec{x}^{-1}$ exist, $\vec{x}^{-1}$ is the restriction to the graph of projection $(x,y,z) \to (x,y)$

$\therefore \vec{x}^{-1}$ is cont.

③ check $d\vec{x}_p$ is non-singular

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0.$$

$\vec{X}_1 = (u, v, \sqrt{1-u^2-v^2})$ $\quad f(u,v) = \sqrt{1-u^2-v^2}$

$U = \{(u,v) \mid v^2+u^2 < 1\}$ $\quad f: U \to \mathbb{R}$.

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Def: Given a differentiable map $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$. We say $p \in U$ is a critical point of F is a differentiable $dF_p: \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective map.

The image $F(p) \in \mathbb{R}^m$ of a critical value of F

$f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$

$\{e_1, e_2, e_3\}$ is a bases of $\mathbb{R}^3$, $e_1 = \{1,0,0\}$ $e_2\{0,1,0\}$, $e_2 = \{0,1,0\}$
$e_3 = \{0,0,1\}$

$df(e_1) = f_x$

$df(e_2) = f_y$

$df(e_3) = f_z$

$a \in f(U)$ is called a regular value of f, iff f_x, f_y and f_z do not vanish

simultaneously at any point in the inverse image
$f^{-1}(a) = \{(x,y,z) \in U \mid f(x,y,z) = a\}$

Prop: If $f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and a is a regular value of f $\left(\frac{df}{dt}(a) \neq 0\right)$, $(p \in f^{-1}(a),\ df_p \neq 0)$

Then $f^{-1}(a)$ is a regular surface

(Hope $f^{-1}(a) \cong$ the graph of a diff. fun)

Ex:

$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$ (S^2 center $(0,0,0)$ radius 1)

Let $f(x,y,z) = x^2+y^2+z^2$

f is C^∞

$f_x = 2x$ — ① ① = 0

$f_y = 2y$ — ② ② = 0 $\Leftrightarrow$ $x=0$

$f_z = 2z$ — ③ ③ = 0 $y=0$

$z=0$

$(0,0,0)$ is a critical point of f

0 is a critical value of f

$(0,0,0) \notin f^{-1}(1) = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

1 is a regular value of f

By prop 2, $f^{-1}(1) = S^2$ is a regular surface

<u>Thm</u>: <u>Inverse Function Theorem</u>

Let $F: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable mapping and suppose that at $p \in U$ the differential map $dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is a isomorphism (i.e $\det(dF)_p \neq 0$) Then $\exists$ Nbhd V of p in U and a Nbhd W of $F(p)$ in $\mathbb{R}^n$

$\ni F: V \to W$ has differential inverse $F^{-1}: W \to V$)

($F: V \to W$: diffeomorphism)

"pf" (of prop.)

Let $p = (x_0, y_0, z_0) \in f^{-1}(a)$, where a is a regular value of f.

We may assume $\frac{\partial f}{\partial z}(p) \neq 0$.

Difine $F: U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.

$(x,y,z) \to (u,v,t)$

$$F(x,y,z) = (x, y, f(x,y,z)) \Rightarrow \begin{array}{l} u = x \\ v = y \\ t = f(x,y,z) \end{array}$$

$$\det(dF_p) = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix} = f_z |_p = \frac{\partial f}{\partial z}(p) \neq 0 .$$

recall : $F: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$, $p \in U$

$dF_p : U \to \mathbb{R}^n$ is a isomorphism ($\det(dF_p) \neq 0$)

$\Rightarrow \exists$ nbhd V of p, Nbhd W of $F(p)$

$F: V \to W$ C^∞ and $F^{-1}: W \to V$ C^∞

By I-F-T, $\exists$ a bbhd N of p and a Nbhd W of $F(p)$

$\Rightarrow F: V \to W \subseteq \mathbb{R}^3$ is diff. and F^{-1} exists and is diff.

$\therefore F^{-1}(u,v,t) = (u, v, g(u,v,t))$

$\therefore$ $x = u$

$y = v$

$z = g(u,v,t)$

In particular, the $g(u,v,t)|_{t=a}$ is C^∞ and we call this function

$h(u,v) = g(u,v,t)\ |_{t=a} = g(u,v,a)$

$$F^{-1}(W \cap \{(u,v,t) \mid t=a\}$$

$$= V \cap \{(u,v,g(u),v,t \mid t=a\} = V \cap \{(u,v, f(u,v,t)\}$$

$f^{-1}(a) \cap V$

$\shortparallel$

graph, $h \in \infty$

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By prop 2, $f^{-1}(a) \cap V$ is a regular surface

($\because$ $f^{-1} \cap V \equiv$ graph of h)

Since P is arbitary

$\therefore$ f^{-1} is a regular surface

Qundric Surface

$A x^2 + By^2 + Cz^2 + DX + EY + fz + Gxy + Iyz + Hxz + J = 0$.

$A \equiv$ a constant 4×4 symmetric matrix $(A^t = A)$

$$A = \begin{bmatrix} a_1 & a_5 & a_6 & a_7 \\ a_5 & a_2 & a_8 & a_9 \\ a_6 & a_8 & a_3 & a_{10} \\ a_7 & a_9 & a_{10} & a_4 \end{bmatrix}$$

Qundric Surface S is given by

$S = \left\{ u \in \mathbb{R}^3 \;\middle|\; (1 \;\; u^t) \, A_{4\times 4} \begin{pmatrix} 1 \\ u \end{pmatrix} = 0 \right\}$

u is a vector and $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ column matrix

Let $f: \mathbb{R}^3 \to \mathbb{R}$

$f(u) = (1 \;\; u^t) \, A \begin{bmatrix} 1 \\ u \end{bmatrix}$, $f^{-1}(0) = S$.

Q: is "0" a regalar value of f

A: "Yes" $S = f^{-1}(0)$ is a regular surface

"No" . . . . .

Look at the differential of F.

$dF: \mathbb{R}^3 \to \mathbb{R}$, take $p \in f^{-1}(0)$, $df_p(w) = ?$ w is a vector.

$\because$ α is a regular curve. $\alpha(0) = p$, $\alpha'(0) = w$

$df_p w = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0}$,

$\because (f \circ \alpha)(t) = (1, \alpha(t)^t) A \begin{pmatrix} 1 \\ \alpha(t) \end{pmatrix}$

$$\frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = (0, w^t) A \begin{pmatrix} 1 \\ p \end{pmatrix} + (1, p^t) A \begin{pmatrix} 0 \\ w \end{pmatrix}$$

$$= 2 (1, p^t) A \begin{pmatrix} 0 \\ w \end{pmatrix} = df_p(w)$$

$df_p(w) = 0 \ \forall w$ iff $(1 \ p^t) A = (k \ 0)$

$p \in f^{-1}(0) \Rightarrow k = 0$.

0 is a critical value iff $(1 \ p^t) A = 0$ has solutions.

0 is a regular value iff $(1 \ p^t) A = 0$ has no solution

<u>prop 3</u>: A regular surface is locally graph of a differentiable function.

($p \in S$, $\exists$ a nbhd V of p in S $\ni$ $V \equiv$ the graph of a C^∞ function)

$z = f(x,y)$, $y = h(x,z)$, $x = g(y,z)$)

<u>"pf"</u> S is a regular surface

$\exists \vec{x}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ which is a parametrization of S.

$\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$

Let $q \in \vec{x}^{-1}(p)$

I particular, we may assume $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$.

We consider the projection map $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ as $\pi(x,y,z) = (x,y)$

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Look at $\pi \circ \vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $\pi \circ \vec{x}(u,v) = (x(u,v), y(u,v))$

$\det\left(d(\pi \circ \vec{x})\right)\Big|_q = \dfrac{\partial(x,y)}{\partial(u,v)}\Big|_q \neq 0$. $\therefore \pi \circ \vec{x}$ is an isomorphism

By the I-F-T,

$\exists$ nbhd V_1 of q, nbhd V_2 of $(\pi \circ \vec{x})(q)$

$\therefore \exists\ \pi \circ \vec{x} : V_1 \to V_2$ is diff. and $(\pi \circ \vec{x})$ exist & C^∞

$\therefore (\pi \circ \vec{x})^{-1} : V_2 \subseteq \mathbb{R}^2 \to V_1 \subseteq \mathbb{R}^2$

$(\pi \circ \vec{x})^{-1}(x,y) = (u(x,y), v(x,y))$

$\therefore z(u(x,y), v(x,y))$ is a function on V_2 s.t a piece of the surface can be represented the graph of γ $(x(u,v), y(u,v), z(u,v))$ rewrite it as $(x, y, z(u(x,y), v(x,y)))$

A regular surface is locally graph of a differentiable function

$\vec{x}: U \subseteq \mathbb{R} \longrightarrow S \subseteq \mathbb{R}^3$, $\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_q \neq 0$, $q \in U$

$\pi \circ \vec{x} = \mathbb{R}^2 \to \mathbb{R}^2$, $\pi : \mathbb{R}^3 \to \mathbb{R}^2$

$\det(d(\pi \circ \vec{X}))_q \neq 0 \Rightarrow$ Inverse Function.

$\pi \circ X : V_1 \to V_2$ differeophism

$\pi|_{X^{-1}(V)} = \pi|_{V_1}$, $(\pi \circ X)^{-1}$ is diff .

$(\pi \circ V)^{-1} : V_2 \to V_1$

$(x,y) \quad (u(x,y), v(x,y))$

$V \equiv$ the graph of $z = f(x,y)$

Quadric Surface

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (ellipsoid)

(hypoboloid with 2 sheets)

Ex: hypoboloid

hypoboloid given by $\{(x,y,z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1\} = S$

① check S is a regular surface

② Find the parametrization of S.

Let $f = \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2 - x^2 - y^2$

then , $S = f^{-1}(1) = \{(x,y,z) \mid f(x,y,z) = 1\}$

$f_x = df(e_1) = -2x$ —①

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$f_y = -2y$ — ②

$f_z = 2z$ — ③

~~①, ②, ③ are not zero~~ simultaneously unless $\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$.

~~However~~ $(0,0,0)$ is not point of f.

1 is a regular value of f.

By prop 2. done.

S can be represented as $z = 1 + x^2 + y^2$

~~This surface has two sheets.~~

one is $z = \sqrt{1+x^2+y^2}$

another is $z = -\sqrt{1+x^2+y^2}$

(ii) $f(x,y,z) = \sqrt{1+x^2+y^2} \rightarrow f$ is diff.

$\therefore \vec{x}_1(x,y) = (x, y, \sqrt{1+x^2+y^2})$

$(\vec{x}_2(x,y)) = (x, y, -\sqrt{1+x^2+y^2})$

① diff. v

② homeomorphism

③ $d\vec{x}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{x}{\sqrt{1+x^2+y^2}} & \frac{y}{\sqrt{1+x^2+y^2}} \end{bmatrix}$ $\quad \frac{\partial(x,y)}{\partial(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0.$

$$d\vec{x}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1+x^2+y^2}} & \frac{-y}{\sqrt{1+3y^2+z}} \end{bmatrix} \quad \frac{\partial(x,y)}{\partial(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0 \therefore d\vec{x}_2 \text{ is}$$

non sigular

Fact: A regular surface may not be connected

__recall__: __Def__: A surface may not connected if any 2 of its points can be joined by a continuus curve by s

2-3

surface, differentiable function on S.

<u>Def</u>: S is a regular surface, $F: S \to \mathbb{R}$ is differentiable if any parametrization $\vec{x}$ of S s.t

$F \circ \vec{x}: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentable

<u>prop</u>: (change of parameters), Let $p \in S$, S is a regular surface in $\mathbb{R}^3$

Let $\left.\begin{array}{l}\vec{x}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3 \\ \vec{y}: V \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3\end{array}\right\}$ be two parametrization of S

$\ni p \in \vec{x}(U) \cap \vec{y}(V) = W$.

Then the change of coordinates

$h = \vec{x}^{-1} \circ \vec{y}: \vec{y}^{-1}(W) \to \vec{x}^{-1}(W)$ is a diffeomorphism.

<u>Note</u>: If point p belongs to two coordinate Nbhds with parameter (u,v) and $(\hat{u}, \hat{v})$, it is possible to pass from one of ~~the~~ two pairs of coordinate to the other by a differentiable transformation.

<u>pf</u>: observe that $\vec{x}^{-1} \circ \vec{y}$ is a homeomorphism.

$\therefore$ h is one-one, oneo & cont. h^{-1}: exists and cont.

$h^{-1} = \vec{y}^{-1} \circ \vec{x}$

Let $r \in \vec{y}^{-1}(W)$, $q \in \vec{x}(W)$ $\therefore$ $h(r) = q$

Since $\vec{x}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$ is a parametrization of S.

Assume that $\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u \neq 0$ ($\because$ $d\vec{x}$ is non-singular.

extend $\vec{x}$ to $F : U \times \mathbb{R} \to \mathbb{R}^3$ by $F(u,v,t) = (x(u,v),\ y(u,v),\ z(u,v)+t)$

$\therefore$ F is differentiable.

$$dF(q) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{bmatrix}, \quad \det(dF(q)) = \left.\begin{pmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ z_u & z_v & 1 \end{pmatrix}\right|_q = x_u y_v - x_v y_u \neq 0$$

$\therefore$ F is isomorphism.

By Inverse Function Theorem, F^{-1} exist, F^{-1}: diff.

$\exists$ nbhd U' of q, Nbhd W' of $F(q)$ respectively

$F^{-1} : W' \to U'$ differentiable

Clearly, $F^{-1}|_{W \cap S} = \vec{x}^{-1}$ ($\because$ $F|_{U \times \{0\}} = x^{-1}$)

Hence $h = \vec{x}^{-1} \circ \vec{y}$ is the same as $F^{-1} \circ \vec{y}$ restrict on $y^{-1}(W' \cap S)$

$\therefore$ h is differentiable

$h^{-1} = (y^{-1} \circ \vec{x}) = G^{-1} \circ \vec{x}$

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Def : $f : V \subseteq S \to \mathbb{R}$ is differentiable at $p \in S$. if for some parametrization. of S,

regular surface

$\vec{x} : U \to S \subseteq \mathbb{R}^3$ with $\vec{x}(U) \subseteq V$

$f \circ \vec{x} : U \to \mathbb{R}$ is diff. at $\vec{x}^{-1}(p)$

recall :

Def : Let $f : V \subseteq S \to \mathbb{R}$ be a function f is called differentiable at $p \in V$

↑ regular surface

if for some parametrization of S $\vec{x} : U \subseteq \mathbb{R}^2 \to S$ with $\vec{x}(U) \subset V$

then $f \circ \vec{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is diff at $\vec{x}^{-1}(p)$

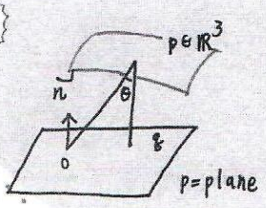
In particular, f is differentiable on V if it's diff. at every point of V.

Observe that if $f : W \subseteq \mathbb{R}^3 \to \mathbb{R}$ is differentiable if $S \subseteq W$

open set of $\mathbb{R}^3$

then $f|_S$ is differentiable $\left(\begin{array}{l} \exists \text{ parametrization } \vec{x} \text{ of } S \\ f \circ \vec{x} \text{ is diff} \end{array} \right)$

Ex: The height function given by a normal direction $\vec{n}$, with $|\vec{n}| = 1$



$f : \mathbb{R}^3 \to \mathbb{R}$

$f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$

$\overline{pq} = \overline{op} \cos\theta = \vec{op} \cdot \vec{n}$

$p \in \mathbb{R}^3$

$f(p) = p \cdot \vec{n}$ diff.

$\overline{pq} = \overline{op} \cos\theta = \vec{p} \cdot \vec{n}$

$p \in S$, $f(p) = p\vec{n}$, height function is diff.

Ex ③ : $f(p) = (p - p_0) \cdot \vec{n}$

Distance square function.

$f : \mathbb{R}^3 \to \mathbb{R}^2$

$d^2 = f(p) = |p-q|^2 = \langle p-q, p-q \rangle \Rightarrow$ diff.

$f(p) = |p-q|^2$ diff.

Ex : ②'

$d = d_q = |p-q| = \langle p-q, p-q \rangle^{\frac{1}{2}}$

$g : \mathbb{R}^2 \setminus \{q\} \to \mathbb{R}$: diffeo.

$S \setminus \{q\} \to \mathbb{R}$ diff

$f : S \to \mathbb{R}$ diff , $f \pm g$: diff , λf diff ,

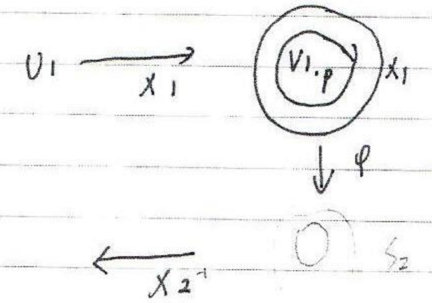
$g : S \to \mathbb{R}$: diff

Cor: A continuous map $\varphi: V_1 \subseteq S_1 \to S_2$ of an open set V_1 of a regular surface S_2 is differentiable at $p \in V_1$ if given two parametrization

$$\left.\begin{aligned} \vec{X}_1 &: U_1 \subseteq \mathbb{R}^2 \to S_1 \\ \vec{X}_2 &: U_2 \subseteq \mathbb{R}^2 \to S_2 \end{aligned}\right\} \text{ with } p \in \vec{X}_1(U_1)$$

and $\varphi(\vec{X}_1(U)) \subseteq \vec{X}_2(U_2)$

$\vec{X}_2^{-1} \circ \varphi \circ \vec{X}_1 : U_1 \to U_2$ is diff. at $q \in \vec{X}_1^{-1}(p)$



Def: Two regular surface are diffeomorphic if a differentiable map $\varphi: S_1 \to S_2$ has a differentiable inverse

$$S_1 \xrightarrow[\text{diffeomorphism}]{} S_2$$

Note: Any regular surface is locally diffeomorphic to an open set in $\mathbb{R}^2$ (i.e) given $p \in S$, $\exists$ nbhd V of p $\ni$ V is diffeomorphic to an open $U \subseteq \mathbb{R}^2$

$\varphi: V \to V$ diffeomorphism

x: ① Let S (regular surface) be symmetric (with respect) w.r to X-Y plane

i.e $(x, y, z) \in S \Rightarrow (x, y, -z) \in S$.

Define $\varphi : S \to S$

$\varphi(x, y, z) = (x, y, -z)$

$\Rightarrow \varphi$: diffeomorphism

② Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation of angle θ about z-axis

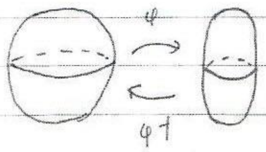
Let $S \subseteq \mathbb{R}^3$ s.t S is invariant under two rotation $R_{z,\theta}$

Let $p \in S$, $\varphi = R_{z,\theta}$ $S \to S$ $\varphi^{-1} = (R_{z,\theta})^{-1} = R_{z,-\theta}$

③ sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$

ellipsoid $E^2 = \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$

where a, b, c are nonzero constants



$\varphi : S^2 \to E^2$

$\varphi(x, y, z) \to (ax, by, cz)$

$\varphi^{-1}(x, y, z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$

φ is diffeomorphism

~~HW: S^2 anti podale map.~~

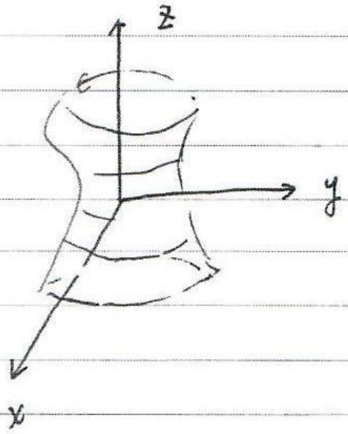
surface of revolution

Take a regular curve in X-Z plane

$C: \mathbb{R} \to \mathbb{R}^3$

$c(v) = (f(v), 0, g(v))$

$\Rightarrow \begin{cases} x = f(v) \boxed{>0} \\ z = g(v) \end{cases} \quad a<v<b$



$\vec{x}: U \subseteq \mathbb{R}^2 \longrightarrow S \subseteq \mathbb{R}^3$

surface obtained by rotating the curve C about the z-axis.

where $U = \{(u,v) \mid 0 < u < 2\pi \ , a<v<b\}$

$\vec{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)\}$

$\Rightarrow$ claim: regular surface

① $\vec{x}$ is differentiable

② $\vec{x}$ is 1-1

$\vec{x}(u,v) = \vec{x}(u',v') \Rightarrow$ $f(v)\cos u = f(v')\cos u'$

$f(v)\sin u = f(v')\sin u'$

$g(v) = g(v') \Rightarrow v = v'$ ($\because$ C: regular curve)

$\Rightarrow \cos u + \sin u = \cos u' + \sin u'$

$\Rightarrow u = u'$

$\Rightarrow \vec{x}$: cont.

$\vec{x}^{-1}$: cont. (HW)

③ check $d\vec{x}$ is non-singular i.e $\frac{\partial \vec{x}}{\partial u} \wedge \frac{\partial \vec{x}}{\partial v} \neq 0$.

$$\frac{\partial \vec{x}}{\partial u} = (-f(v)\sin u,\ f(v)\cos u,\ 0)$$

$$\frac{\partial \vec{x}}{\partial v} = (f'(v)\cos u,\ f'(v)\sin u,\ g'(v))$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -f(v)f'(v) \overset{=0}{}$$

$$\frac{\partial(y,z)}{\partial(u,v)} = g'(v)f(v)\cos u. \quad = 0 \quad \Leftrightarrow \quad \frac{\partial \vec{x}}{\partial u} \wedge \frac{\partial \vec{x}}{\partial v} = 0.$$

$$\frac{\partial(x,z)}{\partial(u,v)} = -f(v)g'(v)\sin u \overset{=0}{}$$

$f'(v) = 0$

$g'(v) = 0,\ \cos u = 0.$

$g'(v) = 0,\ \sin u = 0.$ $\Rightarrow g'(v) = 0.$

$\rightarrow \leftarrow$ $\because$ C: regular curve

$\therefore f'(v) \neq 0,\ g'(v) \neq 0.$